



A LETTERS JOURNAL EXPLORING  
THE FRONTIERS OF PHYSICS

OFFPRINT

**Make slow fast—How to speed up interacting  
disordered matter**

GORAN GLIGORIĆ, KRISTIAN RAYANOV and SERGEJ FLACH

EPL, **101** (2013) 10011

Please visit the new website  
[www.epljournal.org](http://www.epljournal.org)



A LETTERS JOURNAL EXPLORING  
THE FRONTIERS OF PHYSICS

## AN INVITATION TO SUBMIT YOUR WORK

[www.epljournal.org](http://www.epljournal.org)

### **The Editorial Board invites you to submit your letters to EPL**

EPL is a leading international journal publishing original, high-quality Letters in all areas of physics, ranging from condensed matter topics and interdisciplinary research to astrophysics, geophysics, plasma and fusion sciences, including those with application potential.

The high profile of the journal combined with the excellent scientific quality of the articles continue to ensure EPL is an essential resource for its worldwide audience. EPL offers authors global visibility and a great opportunity to share their work with others across the whole of the physics community.

### **Run by active scientists, for scientists**

EPL is reviewed by scientists for scientists, to serve and support the international scientific community. The Editorial Board is a team of active research scientists with an expert understanding of the needs of both authors and researchers.



**IMPACT FACTOR**  
**2.753\***  
\* As ranked by ISI 2010

[www.epljournal.org](http://www.epljournal.org)

## IMPACT FACTOR

# 2.753\*

\* As listed in the ISI® 2010 Science  
Citation Index Journal Citation Reports

## OVER

# 500 000

full text downloads in 2010

# 30 DAYS

average receipt to online  
publication in 2010

# 16 961

citations in 2010  
37% increase from 2007

*"We've had a very positive experience with EPL, and not only on this occasion. The fact that one can identify an appropriate editor, and the editor is an active scientist in the field, makes a huge difference."*

**Dr. Ivar Martin**

Los Alamos National Laboratory,  
USA

## Six good reasons to publish with EPL

We want to work with you to help gain recognition for your high-quality work through worldwide visibility and high citations.

- 1 Quality** – The 40+ Co-Editors, who are experts in their fields, oversee the entire peer-review process, from selection of the referees to making all final acceptance decisions
- 2 Impact Factor** – The 2010 Impact Factor is 2.753; your work will be in the right place to be cited by your peers
- 3 Speed of processing** – We aim to provide you with a quick and efficient service; the median time from acceptance to online publication is 30 days
- 4 High visibility** – All articles are free to read for 30 days from online publication date
- 5 International reach** – Over 2,000 institutions have access to EPL, enabling your work to be read by your peers in 100 countries
- 6 Open Access** – Articles are offered open access for a one-off author payment

Details on preparing, submitting and tracking the progress of your manuscript from submission to acceptance are available on the EPL submission website [www.epletters.net](http://www.epletters.net).

If you would like further information about our author service or EPL in general, please visit [www.epljournal.org](http://www.epljournal.org) or e-mail us at [info@epljournal.org](mailto:info@epljournal.org).

### EPL is published in partnership with:



European Physical Society



Società Italiana  
di Fisica



EDP Sciences

**IOP Publishing**

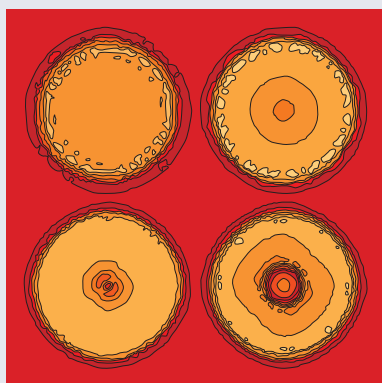
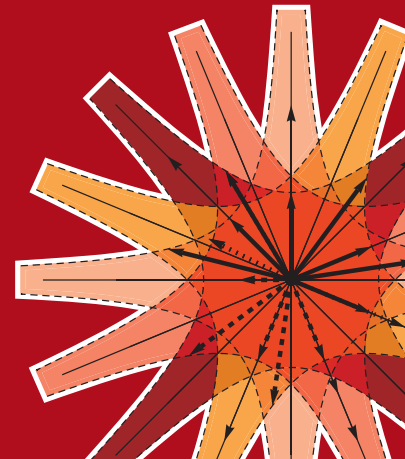
IOP Publishing



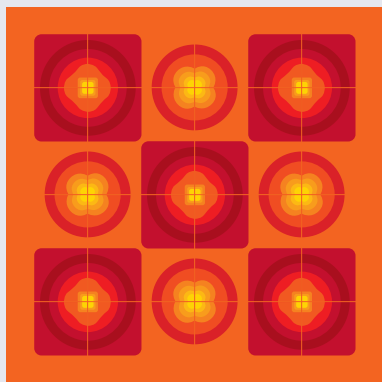
A LETTERS JOURNAL  
EXPLORING THE FRONTIERS  
OF PHYSICS

**EPL Compilation Index**

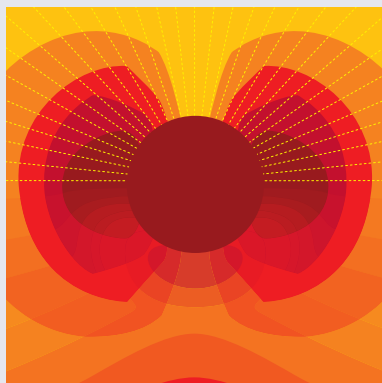
[www.epljournal.org](http://www.epljournal.org)



Biaxial strain on lens-shaped quantum rings of different inner radii, adapted from **Zhang et al** 2008 *EPL* **83** 67004.



Artistic impression of electrostatic particle-particle interactions in dielectrophoresis, adapted from **N Aubry and P Singh** 2006 *EPL* **74** 623.



Artistic impression of velocity and normal stress profiles around a sphere that moves through a polymer solution, adapted from **R Tuinier, J K G Dhont and T-H Fan** 2006 *EPL* **75** 929.

Visit the EPL website to read the latest articles published in cutting-edge fields of research from across the whole of physics.

Each compilation is led by its own Co-Editor, who is a leading scientist in that field, and who is responsible for overseeing the review process, selecting referees and making publication decisions for every manuscript.

- Graphene
- Liquid Crystals
- High Transition Temperature Superconductors
- Quantum Information Processing & Communication
- Biological & Soft Matter Physics
- Atomic, Molecular & Optical Physics
- Bose-Einstein Condensates & Ultracold Gases
- Metamaterials, Nanostructures & Magnetic Materials
- Mathematical Methods
- Physics of Gases, Plasmas & Electric Fields
- High Energy Nuclear Physics

If you are working on research in any of these areas, the Co-Editors would be delighted to receive your submission. Articles should be submitted via the automated manuscript system at [www.epletters.net](http://www.epletters.net)

If you would like further information about our author service or EPL in general, please visit [www.epljournal.org](http://www.epljournal.org) or e-mail us at [info@epljournal.org](mailto:info@epljournal.org)



**IOP Publishing**

**Image:** Ornamental multiplication of space-time figures of temperature transformation rules (adapted from T. S. Bíró and P. Ván 2010 *EPL* **89** 30001; artistic impression by Frédérique Swist).

# Make slow fast —How to speed up interacting disordered matter

GORAN GLIGORIĆ<sup>1,2</sup>, KRISTIAN RAYANOV<sup>1</sup> and SERGEJ FLACH<sup>1,3</sup>

<sup>1</sup> *Max Planck Institute for the Physics of Complex Systems - Nöthnitzer Straße 38, 01187, Dresden, Germany, EU*

<sup>2</sup> *Vinča Institute of Nuclear Sciences, University of Belgrade - P.O. Box 522, 11001 Belgrade, Serbia*

<sup>3</sup> *New Zealand Institute for Advanced Study, Massey University - Auckland, New Zealand*

received 29 November 2012; accepted in final form 2 January 2013

published online 21 January 2013

PACS 05.45.-a – Nonlinear dynamics and chaos

PACS 71.55.Jv – Disordered structures; amorphous and glassy solids

PACS 37.10.Jk – Atoms in optical lattices

**Abstract** – Anderson and dynamical localization have been experimentally observed with ultra-cold atomic matter. Feshbach resonances are used to efficiently control the strength of interactions between atoms. This allows to study the delocalization effect of interactions for localized wave packets. The delocalization processes are subdiffusive and slow, thereby limiting the quantitative experimental and numerical analysis. We propose an elegant solution of the problem by proper ramping the interaction strength in time. We demonstrate that subdiffusion is speeded up to normal diffusion for interacting disordered and kicked atomic systems. The door is open to test these theoretical results experimentally, and to attack similar computational quests in higher space dimensions

Copyright © EPLA, 2013

**Introduction.** – The quantum wave nature of ultra-cold atoms in optical potentials [1], as demonstrated impressively through their macroscopic condensation [2,3], is the key ingredient for the recent observation of Anderson localization with quantum atomic matter [4,5]. Quasi-one-dimensional elongated traps are modulated randomly with speckle potentials [6,7], or simply quasiperiodically with interfering laser beams [8], in order to observe the halt of spreading of an initially localized wave packet of  $10^4$ – $10^5$  Rb and K atoms, and an exponentially localized atomic density distribution profile. The length scales are controlled by the localization length  $\xi$  which is a function of the potential parameters, and possibly also the energies of packet atoms. This phenomenon of wave localization is inherently relying on the phase coherence of matter waves. It is closely related to the dynamical localization of the quantum kicked rotor in momentum space, which was successfully probed already in 1995 using ultra-cold Na atoms [9]. Recent experiments with quasiperiodically kicked rotors with Cs atoms extend to two- and three-dimensional disorder potentials [10]. Interestingly systems of one- and two-dimensional optical waveguides have been also recently used to probe Anderson localization [11,12].

For some atomic species (K, Cs, Na, Li) Feshbach resonances can be used to efficiently control the strength of interactions between atoms [13–16]. This opens the possibility to study the fate of Anderson localization

for interacting localized wave packets. Indeed the first experiment of this kind [17] showed that interaction beats localization, but in a very slow way —the second moment  $m_2$  of an atomic wave packet increases subdiffusively in time:  $m_2 \sim t^\alpha$  with  $\alpha < 1$ . This process may stop in the long run once the atomic density  $n$  of the wave packet reaches the inverse of the localization length  $\xi$  (which touches the quantum world of many-body localization [18]). On shorter times (when typically more than 10 atoms occupy one local single-particle state) the mean-field approximation is a reasonable tool for the study of the subdiffusive process.

**The problem.** – The mean-field approximation replaces the many-body linear Schrödinger equation in a hugely dimensional Hilbert space with a nonlinear Schrödinger equation (NLS), *e.g.*, the Gross-Pitaevsky equation. The effective interaction strength  $\beta$  is proportional to the scattering length  $a_s$ . What matters in terms of quality here, is the fact that for almost any disorder (or quasiperiodic) potential realization the corresponding NLS will be nonintegrable. This seemingly unimportant mathematical property has a very profound impact —the dynamics of a wave packet becomes in general chaotic in time, characterized by positive Lyapunov coefficients and exponential divergence of nearby trajectories. As a consequence the coherence of phases of waves which

constitute a given initial wave packet is lost, and with it also the whole effect of wave localization. First observed in 1993 by Shepelyansky for an NLS version of the quantum kicked rotor [19], it was recently studied with great detail for random and quasiperiodic potentials [20–25]. The main outcome for quasi-one-dimensional models with two-body time-independent atomic interactions is a universal asymptotic wave packet spreading with exponent  $\alpha = 1/3$ , which does not depend on the interaction and disorder strength [26]. With inverse time units equal to single-particle kinetic energies the crossover from intermediate to universal asymptotic dynamics takes place at dimensionless time  $\tau \sim 10^6$ . Although the spreading is induced by interaction, the crossover time between intermediate and universal asymptotic regime of spreading is usually much more affected by the wave packet initial form than by the interaction strength. One exception is the case of very weak interaction, where a long time is needed even for a start of the delocalization process [27]. In the Florence setup [17] largest times reached are  $\tau \sim 10^4$ , leaving the experiments in the intermediate case-to-case-dependent dynamics. Although the onset of subdiffusion is clearly observed, no reliable experimental data are currently at hand to measure the exponent  $\alpha$ , as follows from the data analysis and the large statistical errors in [17]. While some experimental optimization and increase of the kinetic energy may add one order of magnitude in time, another one-two orders are needed and are probably currently out of range of accessibility. Notably similar problems of insufficient available time scales arise with computational studies when turning to higher-dimensional analogs [24,28,29]. While two-dimensional models appear to be at the edge of reasonable analysis, three-dimensional are clearly not. We emphasize here that the issue is not only to reach an asymptotic state, but also how much time is needed to probe this state. As a rule of the thumb any power law needs two orders of magnitude in each of the variables to be diagnosed. A subdiffusive process with  $\alpha = 1/3$  will need six orders of magnitude in time, a normal diffusion process instead only two. As follows from the above we can diagnose the problem of lacking time scales for a safe observation and study of subdiffusive interacting atomic cloud dynamics in disordered media.

**The solution.** – Instead of trying to substantially increase available time scales, we propose here to speed up the subdiffusive process itself. This is done by a temporal ramping of the two-body interaction strength, which can be varied, *e.g.*, for K atoms by three orders of magnitude close to the Feshbach resonance [13]. Why should that help? The momentary diffusion rate  $D$  of a spreading packet in one spatial dimension is proportional to the fourth power of the product of interaction strength  $\beta$  and particle density  $n$ :  $D \sim (\beta n)^4$  [30]. In the course of cloud spreading the density  $n$  decreases, and therefore also  $D$ . This is the reason for the predicted subdiffusion process,

which is substantially slower than normal diffusion. We propose here to compensate the decrease of the density  $n$  with an increase in the interaction strength  $\beta$ . Depending on the concrete time dependence  $\beta(\tau)$  we expect different faster subdiffusion processes, and possibly even normal diffusion. The condition for that outcome to be realized is, that the internal chaos time scales (basically the inverse Lyapunov coefficients) will be still short enough so that the atomic cloud can first get chaotic, and then spread. With that achieved, the cloud spreading will be faster, and we can expect that the available experimental time will suffice for the precise observation and analysis of the process.

Let us get into numbers for one spatial dimension. The second moment is  $m_2 \sim 1/n^2$  and the momentary diffusion constant  $D \sim (\beta n)^4$ . For a constant  $\beta$  the solution of  $m_2 = D\tau$  yields  $m_2 \sim 1/n^2 \sim \tau^{1/3}$ , and therefore  $n \sim \tau^{-1/6}$ . Thus we choose now a time dependence  $\beta \sim \tau^\nu$ . Then the resulting spreading is characterized by

$$m_2 \sim \tau^{(1+4\nu)/3}, \quad d = 1. \quad (1)$$

For  $\nu = 1/2$  we already obtain normal diffusion  $m_2 \sim \tau$ .

Similar for two spatial dimensions, where  $m_2 \sim 1/n$ , for a constant  $\beta$  the cloud spreading is even slower with  $m_2 \sim \tau^{1/5}$ . With a time dependent ramping  $\beta \sim \tau^\nu$  the resulting speedup is

$$m_2 \sim \tau^{(1+4\nu)/5}, \quad d = 2. \quad (2)$$

For  $\nu = 1$  we again obtain normal diffusion.

We note that if numerics confirm the above predictions then also the above conditions for the chaoticity time scales are met with good probability. Returning to the original quantum many-body problem we note that only states within the volume of one localization length do interact. These states show level repulsion and Wigner statistics [31], therefore adiabaticity is a well-defined concept here as well.

Once ramping is too fast, we expect to see several different scenarios. Either fragmenting atomic clouds appear since some parts of the cloud get self-trapped [32,33] and some other parts do not. If self-trapping is avoided, we may also see ramping-induced diffusion: while the internal cloud dynamics does not suffice to decohere phases, initial fluctuations in the density distribution can lead to considerably different temporal energy renormalizations in different cloud spots, and therefore to an effective dephasing similar to a random noise process in real time and space.

**Results in one dimension.** – Here we study the spreading of atomic clouds in one-dimensional disorder potentials and in a quantum kicked rotor with interacting atoms. The first model is described with the discrete NLS (DNLS)

$$i \frac{\partial \psi_l}{\partial \tau} = \epsilon_l \psi_l + \beta(\tau) |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}, \quad (3)$$



in which the on-site energy  $\epsilon_l$  is chosen uniformly from a  $[-W/2, W/2]$  random distribution. The nonlinear quantum kicked rotor (NQKR) is studied within the diagonal interaction approximation introduced by Shepelyansky in [19]:

$$\psi_l(\tau+1) = \sum_m (-i)^{l-m} J_{l-m}(k) \psi_m(\tau) e^{-i\frac{\tau}{2}m^2 + i\beta(\tau)|\psi_m|^2}, \quad (4)$$

where  $\psi_l(\tau)$  are the Fourier coefficients of the corresponding time-dependent many-body wave function.  $J_{l-m}(k)$  is a Bessel function of the first kind, whose argument  $k$  is the kick strength, and  $\bar{\tau}$  is a parameter which relates the period of applied kicks  $T$  (set to  $T=1$ ) to the natural frequency of rotor, defined as  $\omega = \hbar/2M$  ( $M$  is the mass of atoms). In both models  $\beta$  is the interaction strength ramped in time  $\tau$  —the dimensionless time for the DNLS and the number of kicks for the NQKR model:

$$\beta(\tau) = \begin{cases} \beta_0, & \tau \leq \tau_0, \\ \beta_0 \left(\frac{\tau}{\tau_0}\right)^\nu, & \tau > \tau_0. \end{cases} \quad (5)$$

In both models we consider a wave packet which is initially concentrated on a single site for purely technical reasons, without any loss of generality (extended initial clouds are perfectly usable as well [27]). After some first time scale  $\tau_0$  the packet spreads approximately over one localization length  $\xi$ . The total norm of the packet is set to one without any loss of generality and is proportional to the total number of atoms in a cloud (similarly, we could also choose any larger norm and rescale  $\beta$  accordingly). To characterize the spreading of the cloud we compute the density  $n_l = |\psi_l(\tau)|^2$ , the participation number  $P = 1/\sum_l n_l^2$  (the number of strongly excited sites), and the second moment  $m_2 = \sum_l (l - \bar{l})^2 n_l$  (the squared distance between the wave packet tails), where  $\bar{l} = \sum_l l n_l$  is the first moment. In the NQKR the average energy of the atomic cloud is proportional to the corresponding second moment,  $E = \frac{1}{2}m_2$ . Another remarkable difference between both systems is that for large values of  $\beta$  self-trapping can occur for atomic clouds in disordered spatial systems which may lead to soliton formation [34]. For the kicked rotor case this is impossible since the interaction strength  $\beta$  action in (4) is cyclic reflecting the circumstance of periodic kick action in momentum space [35].

Equation (3) was time evolved using a SABA-class symplectic integration scheme and equation (4) as an iteration map. The parameters were fixed to  $\beta_0 = 1$  and  $W = 4$  for the DNLS and  $\beta_0 = 0.4$ ,  $\bar{\tau} = 1$  and  $k = 3$  for the one-dimensional NQKR. Different realizations for the DNLS were produced by choosing different unique random sequences in the interval  $[-W/2, W/2]$ , while for the NQKR they were realized by exciting different initial states.

The spreading of wave packets in the DNLS model, without and with ramping of the nonlinearity are shown in fig. 1. Clearly packets spread faster when the nonlinearity

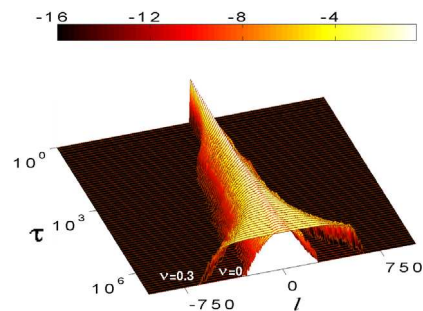


Fig. 1: (Color online) Evolution of the averaged norm density  $\langle n_l(\tau) \rangle$  in the case without ( $\nu = 0$ ) and with ramping ( $\nu = 0.3$ ) in log scale for the DNLS model.

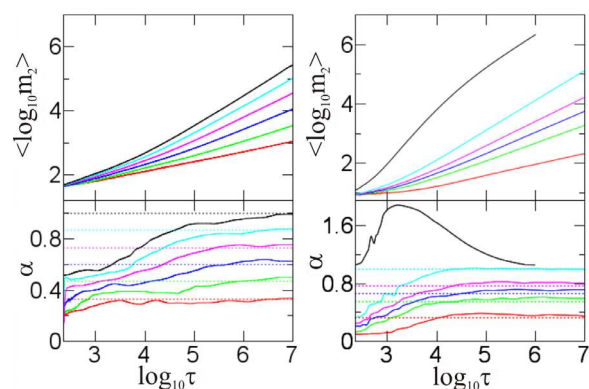


Fig. 2: (Color online) Left column: the second moments (upper) and their power-law exponents  $\alpha$  (lower) for the DNLS model for  $\nu = 0$  (red),  $\nu = 0.1$  (green),  $\nu = 0.2$  (blue),  $\nu = 0.3$  (magenta),  $\nu = 0.4$  (cyan), and  $\nu = 0.5$  (black). Right column: the second moments (upper) and their power-law exponents  $\alpha$  (lower) for the NQKR model for  $\nu = 0$  (red),  $\nu = 0.17$  (green),  $\nu = 0.25$  (blue),  $\nu = 0.33$  (magenta),  $\nu = 0.5$  (cyan), and  $\nu = 1.5$  (black). Dashed colored lines correspond to expected values for exponents in both cases.

is ramped in time. To quantify the spreading exponent, we averaged the logs (base 10) of  $P$  and  $m_2$  over 1000 different realizations and smoothed additionally with locally weighted regression [36]. The (time-dependent) spreading exponents are obtained through central finite difference method [37],  $\alpha = \frac{d(\log_{10}(m_2))}{d(\log_{10}(\tau))}$ . The results for the DNLS and NQKR model are shown in fig. 2. The exponents of subdiffusive spreading reach the theoretically predicted values. Note that the first assumption of the asymptotic exponent occurs after similar waiting times for all  $\nu$ . Monitoring of the participation number  $P$  for the DNLS indicates that self-trapping starts to occur already for  $\nu = 0.4$ . Results for the NQKR model, in which the self-trapping is avoided, confirm the reaching of a normal diffusion process for  $\nu = 0.5$ . Remarkably, the absence of self-trapping for the NQKR results in superdiffusion on intermediate times for  $\nu > 0.5$ . Finally, the exponent relaxes back to the normal diffusion value indicating

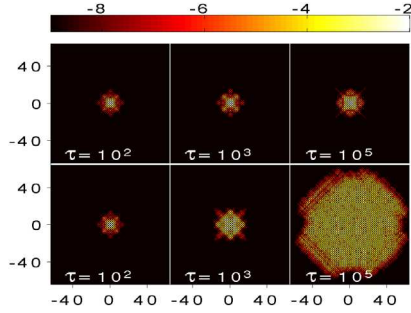


Fig. 3: (Color online) The norm density  $n_{l_1, l_2}(\tau)$  in the case without ( $\nu=0$ , upper row) and with ( $\nu=1.0$ , lower row) ramping after  $\tau=10^2$ ,  $\tau=10^3$ , and  $\tau=10^5$  kicks in log scale for the two-dimensional NQKR model.

the realization of ramping-induced diffusion. This case is illustrated for  $\nu=1.5$  in fig. 2.

**Results in two dimensions.** – To speed up subdiffusive processes in higher dimensions we considered the two-dimensional NQKR model based on the map introduced in [38], with an additional phase term which takes into account interactions in the diagonal approximation:

$$\begin{aligned} \psi_{l_1, l_2}(\tau+1) = & \sum_{s_1, s_2} (-i)^{s_1+s_2} J_{s_1}(k/2) J_{s_2}(k/2) \\ & \psi_{l_1-s_1-s_2, l_2+s_1-s_2}(\tau) \\ & e^{-i\frac{\tau}{2}((l_1-s_1-s_2)^2+(l_2+s_1-s_2)^2)} \\ & e^{i\beta(\tau)} |\psi_{l_1-s_1-s_2, l_2+s_1-s_2}(\tau)|^2. \end{aligned} \quad (6)$$

The notation is the same as in the one-dimensional case, except that now we have two indices, for each possible direction. Note that according to the relation for Bessel functions  $J_{-n}(x) = (-1)^n J_n(x)$ , the wave packet  $\psi_{l_1, l_2}$ , defined by expression (6), exhibits symmetry with respect to the  $l_1$  and the  $l_2$  direction. The density is defined as  $n_{l_1, l_2} = |\psi_{l_1, l_2}(\tau)|^2$ , the participation number as  $P = 1/\sum_{l_1, l_2} n_{l_1, l_2}^2$ , and the second moment is  $m_2 = \sum_{l_1, l_2} [(l_1 - \bar{l}_1)^2 + (l_2 - \bar{l}_2)^2] n_{l_1, l_2}$ , where  $\bar{l}_1 = \sum_{l_1, l_2} l_1 n_{l_1, l_2}$  and  $\bar{l}_2 = \sum_{l_1, l_2} l_2 n_{l_1, l_2}$ . Equation (6) was time evolved as an iteration map for the fixed parameters  $\beta_0 = 0.4$ ,  $\bar{\tau} = 1$  and  $k = 2$ .

In fig. 3 we compare the wave packet evolution for  $\nu = 1$  (normal diffusion) and  $\nu = 0$  at three different moments of time. We clearly observe the symmetry of wave packet and a much more violent spreading in the presence of ramping. The spreading exponents are computed similar to the one-dimensional case. We find very good agreement with the theoretical prediction (fig. 4). Again the asymptotic spreading state is reached faster for stronger ramping.

**Conclusion.** – We have investigated the speeding-up of the subdiffusive spreading in interacting disordered and kicked atomic systems by a proper ramping of the interaction strength in time. We confirm that ramping the interaction strength leads to faster subdiffusion. For

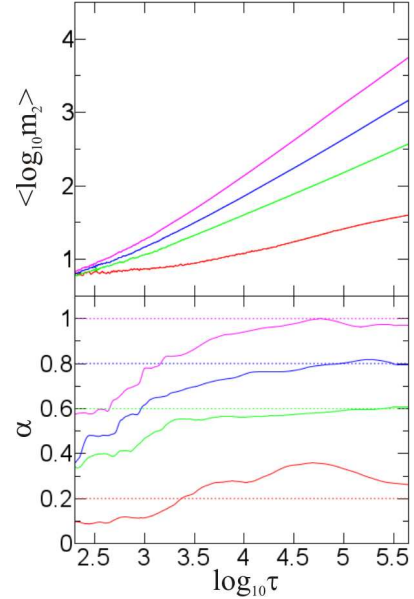


Fig. 4: (Color online) The second moments (upper) and their power-law exponents  $\alpha$  (lower) for the two-dimensional NQKR model for  $\nu = 0$  (red),  $\nu = 0.5$  (green),  $\nu = 0.75$  (blue), and  $\nu = 1$  (magenta). Dashed colored lines correspond to expected values for exponents.

fast enough ramping we even reach normal diffusion of atomic clouds. Self-trapping effects in disordered systems are limiting further speed up of the wave packet spreading. Most importantly the concept works equally well in one-dimensional and two-dimensional systems.

Our results on how to speed up slow subdiffusive processes in interacting disordered matter will be useful for quantitative experimental and computational studies of the impact of interactions on disorder induced matter wave localization. This is particularly true for experimental realizations with ultra-cold atoms, where the scattering length and thus the interactions strength can be tuned via Feshbach resonances by magnetic field variations. On the other hand, our results are also very useful for computational studies of spreading regimes in higher-dimensional systems, where even modern computers reach their limits before reaching the subdiffusive asymptotics.

\*\*\*

GG acknowledges support from the Ministry of Education and Science of Serbia (Project III45010).

## REFERENCES

- [1] BLOCH I., DALIBARD J. and ZWEGER W., *Rev. Mod. Phys.*, **80** (2008) 885.
- [2] ANGLIN J. R. and KETTERLE W., *Nature*, **416** (2002) 211.
- [3] KETTERLE W., *Rev. Mod. Phys.*, **74** (2002) 1131.



- [4] BILLY J., JOSSE V., ZUO Z., BERNARD A., HAMBRECHT B., LUGAN P., CLÉMENT D., SANCHAEZ-PALENCIA L., BOUYER P. and ASPECT A., *Nature*, **453** (2008) 891.
- [5] ROATI G., D'ERRICO C., FALLANI L., FATTORI M., FORT C., ZACCANTI M., MODUGNO G., MODUGNO M. and INGUSCIO M., *Nature*, **453** (2008) 895.
- [6] CHEN Y. P., HITCHCOCK J., DRIES D., JUNKER M., WELFORD C. and HULET R. G., *Phys. Rev. A*, **77** (2008) 033632.
- [7] CLÉMENT D., VARON A. F., RETTER J., SANCHEZ-PALENCIA L., ASPECT A. and BOUYER P., *New J. Phys.*, **8** (2006) 165.
- [8] LYE J. E., FALLANI L., FORT C., GUARRERA V., MODUGNO M., WIERSMA D. S. and INGUSCIO M., *Phys. Rev. A*, **75** (2007) 061603.
- [9] MOORE F. L., ROBINSON J. C., BHARUCHA C. F., SUNDARAM B. and RAIZEN M. G., *Phys. Rev. Lett.*, **75** (1995) 4598.
- [10] CHABÉ J., LEMAIÉ G., GRÉMAUD B., DELANDE D., SZRIFTGISER P. and GARREAU J. C., *Phys. Rev. Lett.*, **101** (2008) 255702.
- [11] LAHINI Y., AVIDAN A., POZZI F., SOREL M., MORANDOTTI R., CHRISTODOULIDES D. N. and SILBERGER Y., *Phys. Rev. Lett.*, **100** (2008) 013906.
- [12] SCHWARTZ T., BARTAL G., FISHMAN S. and SEGEV M., *Nature*, **446** (2007) 52.
- [13] ROATI G., D'ERRICO C., CATANI J., MODUGNO M., SIMONI A., IGNUSCIO M. and MODUGNO G., *Phys. Rev. Lett.*, **99** (2007) 010403.
- [14] WEBER T., HERBIG J., MARK. M., NÄGERL H. and GRIMM R., *Science*, **299** (2003) 232.
- [15] INOUE S., ANDREWS M. R., STENGER J., MIESNER H.-J., STAMPER-KURN D. M. and KETTERLE W., *Nature*, **392** (1998) 151.
- [16] KHAYKOVICH L., SCRECK F., FERRARI G., BOURDEL T., CUBIZOLLES J., CARR L. D., CASTIN Y. and SALOMON C., *Science*, **296** (2002) 1290.
- [17] LUCIONI E., DEISSLER B., TANZI L., ROATI G., ZACCANTI M., MODUGNO M., LARCHER M., DALFOVO F., IGNUSCIO M. and MODUGNO G., *Phys. Rev. Lett.*, **106** (2011) 230403.
- [18] BASKO D. M., ALEINER I. L. and ALTSHULER B. L., *Ann. Phys. (N.Y.)*, **321** (2006) 1126.
- [19] SHEPELYANSKY D. L., *Phys. Rev. Lett.*, **70** (1993) 1787.
- [20] LAPTYEVA T. V., BODYFELT J. D., KRIMER D. O., SKOKOS CH. and FLACH S., *EPL*, **91** (2010) 30001.
- [21] MULANSKY M., AHNERT K. and PIKOVSKY A., *Phys. Rev. E*, **83** (2011) 026205.
- [22] LARCHER M., DALFOVO F. and MODUGNO M., *Phys. Rev. A*, **80** (2009) 053606.
- [23] LARCHER M., LAPTYEVA T. V., BODYFELT J. D., DALFOVO F., MODUGNO M. and FLACH S., *New J. Phys.*, **14** (2012) 103036.
- [24] GARCÍA-MATA I. and SHEPELYANSKY D. L., *Phys. Rev. E*, **79** (2009) 026205.
- [25] GLIGORIĆ G., BODYFELT J. D. and FLACH S., *EPL*, **96** (2011) 30004.
- [26] FLACH S., KRIMER D. O. and SKOKOS CH., *Phys. Rev. Lett.*, **102** (2009) 024101.
- [27] SKOKOS CH., KRIMER D. O., KOMINEAS S. and FLACH S., *Phys. Rev. E*, **79** (2009) 056211.
- [28] LAPTYEVA T. V., BODYFELT J. D. and FLACH S., *EPL*, **98** (2012) 60002.
- [29] MULANSKY M. and PIKOVSKY A., *Phys. Rev. E*, **86** (2012) 056214.
- [30] FLACH S., *Chem. Phys.*, **375** (2010) 548.
- [31] KRIMER D. O. and FLACH S., *Phys. Rev. E*, **82** (2010) 046221.
- [32] KOPIDAKIS G., KOMINEAS S., FLACH S. and AUBRY S., *Phys. Rev. Lett.*, **100** (2008) 084103.
- [33] RODRIGO R. A. and FLACH S., *Phys. Rev. E*, **79** (2009) 016217.
- [34] FLACH S. and WILLIS C. R., *Phys. Rep.*, **295** (1998) 181.
- [35] IZRAILEV F. M., *Phys. Rep.*, **196** (1990) 299.
- [36] CLEVELAND S. W. and DEVLIN S. J., *J. Am. Stat. Assoc.*, **83** (1988) 596.
- [37] HOFFMAN J., *Numerical Methods for Engineers and Scientists* (McGraw-Hill, New York) 1992.
- [38] DORON E. and FISHMAN S., *Phys. Rev. Lett.*, **60** (1988) 867.